UNIT 5:
FORMAL AND INFORMAL LOGIC

Prof. D. José Juan González
Philosophy and Citizenship
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1. INTRODUCTION: LANGUAGE AND ARGUMENTATION

One of the traits that distinguishes human beings from their anthropoid ancestors is the use of language. And one typical characteristic of human language is the use of arguments. An argument, or deduction, is a linguistic segment of some complexity in which, from the position of starting pieces or subsegments, a final piece or subsegment is necessarily deduced. For example:

1. If there is risk of rain, the barometer goes down; but the barometer doesn't go down. Therefore, there is no risk of rain.

The main parts or linguistic units of an argument are called statements. The starting statements of an argument are called premises and the final statement is called conclusion.

But, what is the use of all this? Does examining how humans argue have any sense? Arguments are used both in ordinary and scientific life. Even more, the utility of this linguistic instrument is great, and is related to something that define us as human beings: the exercise of reason. Arguments allow us to go from the acceptance of some statements (the premises) to the acceptance of others (conclusions), using only our reflective power. We overcome the pure level of immediate knowledge and widen somehow, or at least clarify, our information about the world around us.

But there are some implicit problems in all this: how can we be sure that we have reasoned correctly? How can we be sure that when developing an argument we have not gone wrong at some step in the deductive process, with the result that although the premises would be true, the conclusion we have arrived at would be false? There are many ways to examine an argument to be sure of its correction. The simplest one – and also the most difficult one – examines, one by one, the correctness of each argument. But this procedure will lead no way. To be sure, we would also have clear some kind of criteria that will allow us to classify the arguments somehow, so that we will not admit as valid an incorrect argument, and as invalid a correct argument. The science that tries to find out the criteria for the correction of arguments is called Logic, and although at the beginning, the language that it used was very similar to ordinary language; nowadays, the way it proceeds and explains its conclusions and criteria is much more similar to mathematics than to any other science.

Let's examine an example to check how logic proceeds. Besides argument 1
(above), we can consider the following argument:

2. If I want to pass the Logic examination, I will study on Saturday for it; but I
did not study on Saturday for it.
Therefore, I don't want to pass the Logic examination.

At first sight, the two arguments don't have any relation between them. One of
them talks about the rain and the barometer; while the other talks about examinations
and studies. Nevertheless, if we go further than their materiality (what the arguments
state exactly), we will soon realize that both of them are identical: both of them have
exactly the same basic form:

If \( A \) then \( B \); but no \( B \).
Therefore, no \( A \).

\((A\text{ and }B\text{ can be any statements})\)

Observe how the logical form was hidden from us by what each statement said
(its materiality), so that after substituting each statement by a statement variable \( A, B \),
we have found it and have come to know that both arguments are identical from the
logical point of view. But this has another important consequence: if both have the
same logical form, if we find some way to probe the correctness of that form, we will
also know, automatically, that all other arguments with that same logical form will also
be correct.

But the process of formalization (of leaving an argument in its pure logical form)
does not stop here. The logical form that we have discovered above still has elements
that belong to ordinary language: “If..., then, but...”. And ordinary language, which is
perfectly suited for all uses in ordinary life, can, once more, lead us astray. Take the
following argument in consideration:

3. When I arrive on time to class, the door is open; but the door isn't open.
Therefore, I do not arrive on time to class.

If we repeat the same operation we did above, the logical form of this argument,
although very similar to the previous one, will be slightly different:

When \( A, B \); but, no \( B \).
Therefore, no \( B \).

Should we draw the conclusion that we are facing another type of argument? If
we substitute the letters \( A \) and \( B \) in this last case for the statements of any of the
previous arguments, we will see that the sense of the argument does not change. In
other words, the type of argument, its form, must be the same.
To avoid misunderstandings of this kind, the science of Logic decided to abandon ordinary or natural language, and to construct a completely artificial language with no traces of that language. This implies, therefore, the substitution of all those particles that are not, properly speaking, statements, by some kind of symbols (i.e. If... then, but...), and also establishing the meaning or sense of such symbols, in such a way that each logical symbol would represent all those logical connections of ordinary language which have the same meaning, no matter the words that one could use to express such logical connections. In our example, and following this analysis, we can see that “If... then...” and “when... then...” are identical logical connections. This means that we should symbolize them with the same element. And then, the final logical form of the arguments that we have used as examples is the following:

\[(p \rightarrow q) \land \neg q \rightarrow \neg p\]

The intention of the first part of this unit is help you understand this language.
2. SYMBOLIC LOGIC: PROPOSITIONAL LOGIC

2.1. Statements and types of statements

1. Definition: A statement or proposition is a segment of language which is either true or false.
   For example: “Sunday afternoons remind me of Mondays”, “Yesterday afternoon, I went to the cinema”

2. Types of propositions: There are two types of propositions:
   a) Atomic propositions: Those in which there is no logical link or connector.
      Ex.: “Einstein developed the theory of relativity.”
      “Seafood makes an exquisite dish.”
   b) Molecular propositions: Those united by one or more logical connectors or links; those formed by the union of two or more atomic propositions by the use of logical connectives.
      Ex.: “If we go to Torrevieja, then we will take a bath in the Cura Beach.”. This is a molecular proposition formed by two atomic propositions: “We go to Torrevieja” and “We will take a bath in the Cura Beach”, joined by the logical connective: “If... then...”.

2.2. Formalization of propositions

1. Definition: Formalization is the operation by which expressions in ordinary language are translated into symbolic expressions of formal language.

2. Steps: To formalize one must:
   a) Substitute the statements or propositions for propositional variables.
   b) Substitute the links that join the statements – if there are any – for logical constants.

2.3. Vocabulary of the propositional symbolic logic

The vocabulary of propositional logic is composed of symbols. There are two types of symbols in this logic:

1. LOGICAL VARIABLES: are the symbols that represent statements. They are also called statements variables or propositional variables. They are symbolized using the letters: p, q, r, s, t, u. (v is not used because it can be
confused with the symbol used for the logical disjunction). When there is a need of more variables, we can start from \( j \).

Ex: “Jaime is stubborn” = \( p \)

“Snow is white” = \( q \)

2. **LOGICAL CONSTANTS**: are the logical symbols that represent the particles that, with the exception of the negation, join atomic propositions in a molecular proposition. They are also called logical connectives.

The logical constants are five:

1. **NEGATION**: It translates those expressions of ordinary language that mean: “not” “neither”, “it is false that...” “It is not true that...”. *It is read* “not” or “It is not the case that...”. It is symbolized by: \( \neg \)

Ex: If the atomic statement “There is a beach in Torrevieja” is \( p \), then “There is no beach in Torrevieja” will be \( \neg p \).

2. **LOGICAL CONJUNCTION**: It translates those expressions of ordinary language that mean: “and”, “although”, “but”, “nevertheless”... *It is read* “and”. It is symbolized by: \( \land \)

Ex.: If “John went for a walk” is \( p \), and “John met Mary” is \( q \), then “John went for a walk and met Mary” will be \( p \land q \).

3. **LOGICAL DISJUNCTION**: In logic there are two types of disjunctions:

   a) **INCLUSIVE DISJUNCTION**: It translates those expressions of ordinary language that mean: “or”. It must be observed that in the correctly formed expressions of ordinary language of this type, two or more expressions are connected, but the true of one of them does not exclude the true of the other. *It is read* “or”. It is symbolized by: \( \lor \)

   Ex.: If “Anthony is a baker” is \( p \), and “Anthony is a football-player” is \( q \), then “Anthony is a baker or a football-player” will be \( p \lor q \). It must be observed that, in this disjunction, the fact of Anthony being a baker does not exclude the other possibility: that he would be a football-player as well. This shows that it is an inclusive disjunction.

   b) **EXCLUSIVE DISJUNCTION**: It translates those expressions of ordinary language that mean: “either... or...”, “or... or...” … *It is read* “exclusive or”. It is symbolized by”: \( \oplus \)

   Ex.: If “Mary was born in San Vicente” is \( p \), and “Mary was born in El Campello” is \( q \), then “Either Mary was born in San Vicente or in El Campello” will be \( p \oplus q \)
It must be observed that, in this disjunction, the truth of one of the two propositions excludes necessarily the truth of the other; in our example, it is impossible that Mary would have been born in San Vicente and in El Campello — if it is the same person. This shows that it is an exclusive disjunction.

4. **CONDITIONAL OR IMPLICATION:** It translates those expressions of ordinary language that mean: “if..., (then)...”, “when..., (then)...” as well as any other expressions that means some type of condition. It is read “if... then...” or “... implies ...”. It is symbolized by: $\rightarrow$

Ex.: If “I want to pass Philosophy” is $p$, and “I must study logic” is $q$, then “If I want to pass Philosophy, then I must study logic” will be $p \rightarrow q$

NOTE: The statement that comes before the conditional symbol is called **antecedent** while the one that comes after the symbol is called **consequent**. It could happen that in the ordinary language expression that we are trying to formalize the antecedent would be after the consequent. Nevertheless, when formalizing, the antecedent – which expresses the condition – must always be placed before the symbol, while the consequent – expressing the consequence – must always be placed after the symbol.

Ex.: The proposition “I will bring you flowers, if I go to Valencia”, being $p$ “I will bring you flowers” and $q$ “I go to Valencia” will be symbolized as $q \rightarrow p$, because the condition for bringing flowers is the trip to Valencia.

5. **BICONDITIONAL OR EQUIVALENCE:** It translates those expressions of ordinary language that mean: “If and only if..., (then)...”, “when and only when..., (then)...” “... is equivalent to ...” “... is equal to ...” … It is read “if and only if..., then...”. It is symbolized by: $\leftrightarrow$

Ex.: If “I want to go to Madrid” is $p$, and “I will go by train” is $q$, then “If and only if I want to go to Madrid, I will go by train” will be $p \leftrightarrow q$

### 2.4 Use of parentheses

1. **Double function of parentheses:** The use of parentheses has a double function in logic formalization:
   a) It shows the influence or range of each logical connective.
   b) It shows which logical connective is dominant over the rest in a given logical formula.

2. **Rules for a correct use of parentheses:**
   a) The **negation** doesn't need parentheses whenever is applied to an atomic proposition, but it does need it if applied to a molecular proposition:
      Ex.1: “I don't want him to see me”
If \( p = \text{“I want him to see me”} \) then \( \neg p \)

Ex.2: “It is false that I want him to see me and to smile at me”
\( p = \text{I want him to see me} \)
\( q = \text{I want him to smile at me} \)

\( \neg(p \land q) \)

b) The **conjunction** doesn't need parentheses in a molecular proposition formed only by conjunctions. The same happens with both **disjunctions**.

Ex.1: “I want a Coke, a Sprite and a cake”
\( p = \text{I want a Coke} \)
\( q = \text{I want a Sprite} \)
\( r = \text{I want a cake} \)

\( p \land q \land r \)

Ex.2: “I want a Coke or a Sprite or a cake”
\( p \lor q \lor r \)

c) When in a molecular proposition there is more than one type of connective, we must use parentheses to limit the reach of each connective, indicating which atomic propositions each connective joins. This way we will avoid misunderstandings.

Ex.1: “Either they give me the Nobel Prize and the Cervantes or I will get upset and won't eat during a whole month”
\( p = \text{they give me the Nobel Prize} \)
\( q = \text{they give me the Cervantes} \)
\( r = \text{I will get upset} \)
\( s = \text{I will not eat during a whole month} \)

\( (p \land q) \lor (r \land \neg s) \)

Ex.2: “If you go to school and forget your books, you will be punished”
\( p = \text{you go to school} \)
\( q = \text{you forget your books} \)
\( r = \text{you will be punished} \)

\( (p \land q) \rightarrow r \)
# LOGICAL CONNECTIVES

<table>
<thead>
<tr>
<th>CONNECTIVE</th>
<th>It translates...</th>
<th>It is read...</th>
<th>Example</th>
<th>Formalization</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Negation</strong> ¬</td>
<td>Not; neither; it is false that...; It is not true that...</td>
<td>Not It is not the case that...</td>
<td>There is no beach at Torrevieja</td>
<td>¬ p</td>
</tr>
<tr>
<td><strong>Conjunction</strong> ∧</td>
<td>and, although, but, nevertheless...</td>
<td>And</td>
<td>John went for a walk and met Mary</td>
<td>p ∧ q</td>
</tr>
<tr>
<td><strong>Inclusive Disjunction</strong> ∨</td>
<td>Or</td>
<td>Or</td>
<td>Anthony is baker or football-player</td>
<td>p ∨ q</td>
</tr>
<tr>
<td><strong>Exclusive Disjunction</strong> w</td>
<td>either... or...; or... or...; ...</td>
<td>Exclusive or...</td>
<td>Either Mary was born in San Vicente or she was born in San Juan</td>
<td>p w q</td>
</tr>
<tr>
<td><strong>Conditional or Implication</strong> →</td>
<td>if..., (then)...; when..., (then)...; any other expressions that means some type of condition.</td>
<td>If..., then... ... implies that...</td>
<td>If I want to pass Philosophy, I must study Logic</td>
<td>p → q</td>
</tr>
<tr>
<td><strong>Biconditional or equivalence</strong> ↔</td>
<td>If and only if..., (then)...; when and only when..., (then)...; ... is equivalent to ...; ... is equal to ...; ...</td>
<td>If and only if..., then...</td>
<td>If and only if I want to go to Madrid, I will catch the train</td>
<td>p ↔ q</td>
</tr>
</tbody>
</table>
3. RULES FOR THE TRANSFORMATION OF FORMULAE AND LOGICAL DEDUCTION

The previous explanation is only a first step towards logical deduction. We said then that Logic intends to offer us the criteria to know whether an argument is correct or not. Thus, we can say that an argument is correct when it follows the rules that we are going to describe below, and incorrect when it doesn’t.

Such criteria are expressed as rules for the transformation of formulae. The meaning of this set of rules is easy to understand although learning how to apply them might take some time. Once an argument has been formalized, we have to prove that the conclusion can be deduced from the premises. And to prove it, Logic only trusts in a set of basic rules whose security or certitude is based on their simplicity. They would be the axioms of a system whose truth must be presupposed because they are based on the basic way human beings reason.

These rules are, truly, rules for the transformation of formulae. Applying them, we actually transform or change the initial statements or premises into others until we get to the proposition that works as the conclusion of our argument. If we achieve this end, the argument will be correct because it has been possible to reach to the conclusion basing or reasoning, at every step, in basic rules of which we are completely assured. If it is impossible to prove it, the argument will not be correct.

The rules for the transformation of formulae allow us, therefore, to go from one formula to another one by their application. This is a fundamental step when deducing logically.

1. *Definition*: A deduction or logical derivation is a set of propositions that are, or a premise (1), or a proposition introduced by a rule for the transformation of formulae (RTF) (2), or the conclusion, which is the last proposition (3).

2. *Elements*: The elements of the logical deduction are the following:

   (a) *Suppositions*. There are two kinds:
   
   i. *Initial suppositions or premises*: the given propositions from the beginning of the deduction.

   ii. *Subsidiary or provisional suppositions*: the propositions introduced provisionally and that must be cancelled or discharged before arriving at the conclusion. They are freely introduced according to their utility to
reach the conclusion.

(b) **Inferred or deduced lines or propositions** from other line or lines applying RTF.

(c) **Rules for the transformation of formulae**, that allow us to construct a new proposition from a previous one or ones.

(d) **Conclusion** or last proposition. It is the proposition, atomic or molecular, that we intend to prove that it can be logically deduced from the previous propositions.

A logical deduction or derivation can be described as a system of lines, consecutively numbered, in which, beginning from a series of starting lines or premises, we try to deduce or arrive at a proposition or conclusion, which will be the last line of our deduction.

To construct a formal deduction we will proceed in the following way:

We will write, **in first place**, and numbering them, the *premises or initial suppositions*, if any, marking them with a P at the right of the formula, as shown below. We will also write the *conclusion* to the right of the last line where we have written our last initial supposition. To symbolize that the formula is the conclusion, we will use the symbol ⊢, called *turnstile*, and read “therefore...” Thus, if we are asked to deduce the conclusion s from the premises, \((p ∧ t) → (r ∧ s), q → t, q ∧ j, y p\), then we will write:

1. \((p ∧ t) → (r ∧ s)\)  
2. \(q → t\)  
3. \(q ∧ j\)  
4. \(p\)  
\[\begin{array}{l}
\vdash s
\end{array}\]

**Secondly**, we will infer, if possible, *new lines* aimed at reaching at the conclusion. To do it, we will apply the rules for the transformation of formulae – that we will learn later on -, in first place, to the initial premises, and after to the new lines deduced or derived from those premises. To show what rule we are using, we will write to the left of the line resulting from its application, the abbreviation of the rule followed by the number or numbers of the preceding derivation line or lines where we have the propositions that justify the application of that rule. If we continue with the previous example, we will get the following:

1. \((p ∧ t) → (r ∧ s)\)  
\[\begin{array}{l}
\vdash s
\end{array}\]
2. $q \rightarrow t$  
3. $q \land j$  
4. $p$  
5. $q$  
6. $t$  
7. $p \land t$  
8. $r \land s$  
9. $s$  

$P$  
$P$  
$P \rightarrow s$  
$RE \land 3$  
$RE \rightarrow 2, 5$  
$RI \land 4, 7$  
$RE \rightarrow 1, 7$  
$RE \land 8$

NOTE: As it is shown in the example, to the right of each formula and in a column (lines 5 - 9), we have written the abbreviation of the rule that we have applied to deduce that logical formula. For example, to the left of line 5, we have written $RE \land 3$. This means that we have deduced line 5. $q$; applying the Rule of Elimination of the Conjunction on line 3. It must be observed, moreover, that each new line deduced from the previous ones can be used to deduce a new line, either by itself or with some other line already deduced or supposed, in order to get to the conclusion. The reason for this is that it is understood that the new line correctly deduced from the previous one is a proposition as true as also are true the propositions from which it has been deduced, so that we can add the new line/proposition to the ones we already have.

In last place, it must be remembered that the purpose of this example is only to show the general way of proceeding in logical deduction. It is not necessary, for now, to understand each step, which is something that will be understood after explaining each of the rules for the transformation of formulae.

There are, in propositional logic, different systems of rules for the transformation of formulae, which are distinguished according to the purpose of their author. Thus, for example, Russell and Whitehead, in their *Principia Mathematica*, proposed a basic list of three rules, because they intended to formalized with them and to construct a complete mathematical system. We are going to study a system of rules called, **Natural Deduction**. It was composed by Gerhard Karl Erich Gentzen, around 1934, and it is considered clearer and more intuitive than the axiomatic deductive systems. We will only study the **basic rules** of this system.

These basic rules are divided into two types: rules of *introduction* and rules of *elimination* of connectives. Given that the basic connectives are only four (¬, ∧, ∨, →), and that we have two rules for each connective, there will therefore be a total of eight basic rules. In the explanation of each rule, we will find a logical scheme of the rule, followed by an explanation of what the rule allows us to do. It must be taken into
account that the variables (letters) that appear in those schemas can be substituted by any propositional formula (that is, both by atomic propositions (ex.: p) and by molecular propositions (ex.: p ∧ q)). We will start by the easiest ones to understand and to use, although that will mean not following their logical order.
1. Rule of the Elimination of the Conjunction (RE ∧)

\[
\frac{A \land B}{A} \quad \frac{A \land B}{B}
\]

This rule allows us to eliminate the conjunction of a given formula and states that if in a line of a deduction we have a conjunction of two propositions, \( A \land B \), we can write any of its members alone in a new line.

2. Rule of the Introduction of the Conjunction (RI ∧)

\[
\frac{A}{B} \quad \frac{B}{(A \land B)}
\]

This rule allows us to introduce the conjunction into a formula, and states that if in a line of a deduction we have a proposition \( A \), and in another line we also have the proposition \( B \), then we can write its conjunction, \( A \land B \), in a new line.

3. Rule of the Introduction of the Disjunction (RI ∨)

\[
\frac{A}{(A \lor B)} \quad \frac{B}{(A \lor B)}
\]

This rule allows us to introduce the disjunction, and states that if in a line of a deduction we have a proposition \( A \), then we can write in a new line that same proposition \( A \) joined by the disjunction to any other proposition \( B \), no matter if that other proposition \( B \) was or wasn’t in the previously deduced lines or in the initial suppositions: \( A \lor B \).

4. Rule of the Elimination of the negation (RE ¬)

\[
\neg \neg A \quad \frac{\neg \neg A}{A}
\]

This rule allows us to eliminate the negation, and states that if in a line of a deduction we have a proposition with a double negation, \( \neg \neg A \), then we can write in a new line that same proposition asserted, \( A \).
5. Rule of the Elimination of the Implication (RE→)

\[
\begin{array}{c}
A \rightarrow B \\
A \\
B
\end{array}
\]

This rule allows us to eliminate the implication, and states that if in a line of a deduction we have an implication, \( A \rightarrow B \), and in another line we have the antecedent, \( A \), of that implication, we can write its consequent in a new line, \( B \).

6. Rule of the Introduction of the Implication (RI→)

\[
\begin{array}{c}
A \\
\vdots \\
B \\
A \rightarrow B
\end{array}
\]

This rule allows to introduce the implication, and states that if in a line of a deduction we introduce a supposition, \( A \), of which we derive the conclusion \( B \) in another line, then we must write \( A \rightarrow B \) in a new line.

**GENERAL NOTE ON SUBSIDIARY OR PROVISIONAL SUPPOSITIONS**

We must remember now that a provisional supposition is a proposition introduced provisionally which must be “cancelled” or “discharged” before reaching the conclusion. They are introduced at will, by convenience, to reach the conclusion. Their use is subject to a series of basic rules intended to avoid erroneous deductions after their employment.

In this sense, and as it is possible to see in the schema of the Rule of the Introduction of the Implication, subsidiary suppositions are marked in a special way: we must use a big square bracket to the left of the formula, starting in the line where we introduce the subsidiary supposition, and ending in the line where that suppositions is cancelled. Besides, we will write an \( S \) to the write of the supposed formula, in the same way that before we wrote the name of the Rule of Transformation we were using. The sense of the big bracket is to underline the fact that in that point we are starting a logical sub-derivation, a sort of sub-set inside the greater set, which follows some special rules:

1. The formulae or lines of deduction derived inside the bracket cannot be used outside of it to get any other formula. Nevertheless, we are allowed to use any formula outside the bracket to deduce a new line inside of it. For example, the following deduction would be incorrect because it goes
against this rule:

1) \( p \rightarrow (q \land r) \)  
2) \( r \rightarrow s \)  
3) \( p \)  
4) \( q \land r \)  
5) \( r \)  
6) \( s \)  
7) \( p \rightarrow s \)  
8) \( q \) 

\[ \begin{align*}
3) & \quad (q \land r) \quad \text{RE } 1, 3 \\
5) & \quad r \quad \text{RE } 4 \\
6) & \quad s \quad \text{RE } 2, 5 \\
7) & \quad p \rightarrow s \quad \text{RI } 3-6 \\
8) & \quad q \quad \text{RE } 4 \\
\end{align*} \]

Line 8 is wrongly deduced because we have derived it from line 4 which is inside of the logical sub-derivation started in line 3, where we have supposed \( p \). The following deduction is correct because it respects this rule:

1) \( p \rightarrow (q \land r) \)  
2) \( r \rightarrow s \)  
3) \( p \)  
4) \( q \land r \)  
5) \( q \)  
6) \( r \)  
7) \( s \)  
8) \( q \land s \)  
9) \( p \rightarrow (q \land s) \) 

\[ \begin{align*}
3) & \quad (q \land r) \quad \text{RE } 1, 3 \\
5) & \quad q \quad \text{RE } 4 \\
6) & \quad r \quad \text{RE } 4 \\
7) & \quad s \quad \text{RE } 2, 6 \\
8) & \quad q \land s \quad \text{RI } 5, 7 \\
9) & \quad p \rightarrow (q \land s) \quad \text{RI } 3 \rightarrow 8 \\
\end{align*} \]

2. The cancellation of the supposition is done in the line immediately following the last line of the supposition sub-set. It is annotated writing the rule that we are applying to close it (in our example, RI \( \rightarrow \)), followed by two numbers separated by a dash. The first number refers to the line-number where the supposition was started, while the second one refers to the line-number where the supposition ends. Thus, in the previous example, we have written RI \( \rightarrow \) 3 – 8, in line 9, because it is there where the supposition is finally cancelled; number 3 refers to line 3 where the supposition was opened; and number 8 refers to line 8 where the supposition is closed.

3. Any subsidiary supposition introduced in a logical deduction must be cancelled out before ending the deduction. Whenever two or more subsidiary suppositions are introduced, the brackets that cancelled them out cannot cut each other. The following diagram shows which schemas are correct and which ones are not:
4. For example, the following deduction is correct for respecting this rule:

1) \((p \land r) \rightarrow q\)  
P \[ \vdash p \rightarrow (r \rightarrow q) \]
2) \(p\)  
S
3) \(r\)  
S
4) \(p \land r\)  
RI \(\land\) 2, 3
5) \(q\)  
RE \(\rightarrow\) 1, 4
6) \(r \rightarrow q\)  
RI \(\rightarrow\) 3 – 5
7) \(p \rightarrow (r \rightarrow q)\)  
RI \(\rightarrow\) 2 – 6

But, the following deduction is incorrect for not respecting this rule in two senses: on the obligation to cancel the supposition before ending the deduction, and on the prohibition that the brackets of different suppositions cannot cut one another.

1) \(p \rightarrow q\)  
P
2) \(q \rightarrow s\)  
P
3) \(s \rightarrow t\)  
P \[ \vdash p \rightarrow t \]
4) \(p\)  
S
5) \(q\)  
S
6) \(s\)  
RE \(\rightarrow\) 2, 5
7) \(t\)  
RE \(\rightarrow\) 3, 6
8) \(p \rightarrow t\)  
RI \(\rightarrow\) 4 – 7
7. Rule of the Elimination of the Disjunction (RE ∨)

This rule allows to *eliminate the disjunction*, and states that if in a line of a deduction we have a disjunction, $A \lor B$, and in another line we introduce the supposition $A$, first term of the disjunction, from which we deduce the conclusion $C$, and after, in another line we introduce the supposition $B$, second member of the same disjunction, from which we derive the same conclusion $C$, then we can write $C$ in a new line, outside now of both sub-sets of suppositions.

As it can be observed, this rule allows us to work with the disjunction, provided that supposing both terms of the disjunction we arrive to the same conclusion. To use it correctly, we must avoid the following common errors:

1. We must remember the special rule for suppositions number 1. That is, the formulae or lines that we derive supposing the first term of the disjunction cannot be used in the sub-derivation started by the second term of the disjunction.

2. To distinguish the suppositions of this type (which are terms of a disjunction) from the suppositions of other types, it is convenient to mark them in a special way. Thus, we will write in parentheses the word (Cas), which is the abbreviation of Case Proof, after the S of supposition, to the right of the supposed formula. In this way, it will be easier to remember that, in order to cancel this type of supposition and end the application of this rule, there must be as many suppositions (Cas) as terms or formulae composing our disjunction.

3. The cancellation of the suppositions introduced by the rule of the elimination of the disjunction is to be done at the end of the whole Case Proof, and in the line immediately posterior to the cancellation of the last supposition; that is, once that we have supposed every term of the disjunction with which we are operating, and have reached, in each occasion, the same conclusion. When writing the cancellation of the supposition, we will annotate the abbreviation of the rule used followed by two numbers separated by a dash. The first number refers to the line where we supposed the first term of the disjunction, while the second one refers to the last line of the supposition we just have closed:
Example of the correct application of **RE ∨**:

1) \( p \lor q \)  
2) \( p \rightarrow r \)  
3) \( r \rightarrow (t \land s) \)  
4) \( q \rightarrow m \)  
5) \( m \rightarrow t \)  
6) \( p \)  (S Cas)  
7) \( r \)  (RE \( \rightarrow \) 2, 6)  
8) \( t \land s \)  (RE \( \rightarrow \) 3, 7)  
9) \( t \)  (RE \( \land \) 8)  
10) \( q \)  (S Cas)  
11) \( m \)  (RE \( \rightarrow \) 4, 10)  
12) \( t \)  (RE \( \rightarrow \) 5, 11)  
13) \( t \)  (RE 6 - 12)

As it can be observed, we have written *Cas* after the S of supposition (lines 6 and 10). Moreover, the first supposition (lines 6 to 9) closes without any special annotation; we simply suppose the next term of the disjunction in line 10. In last place, when we have reached the same conclusion also in the second supposition (line 12), then we cancel that supposition as well, and we write the conclusion we have reached supposing both terms of the disjunction once again (line 13). To annotate it, we write the abbreviation of the rule followed by two numbers separated by a dash: number 6, because there we supposed the first term of the disjunction; and number 12, because there ends all the sub-derivation.

8. **Rule of the Introduction of the Negation** (*Reductio ad absurdum*) (**RI \( \neg \)**)

\[
\begin{array}{c}
A \\
\vdots
\hline
B \land \neg B \\
\hline
\neg A
\end{array}
\]

This rule allows to *introduce the negation*, and states that if in a line of a deduction we introduce a supposition, \( A \), from which we infer as a conclusion a contradiction, \( B \land \neg B \), in another line; then we must write the negation of that supposition, \( \neg A \), in a new line.

This is, besides, a special rule, because by its use it is possible to deduce any possible conclusion, as long as the conclusion be deducible from the given premises.
Nevertheless, it should never be used as a first option, because if it is possible to reach the conclusion using any of the other seven rules or their combination, applying the RI \( \neg \) will mean having to develop a much more complicated deduction.

To apply it correctly, we must first understand its sense. With it, it is possible to deduce any formula as long as from its negation a contradiction follows. Thus, for example, if I want to deduce formula \( A \), we must start supposing its opposite, \( \neg A \), and then try to deduce in that sub-derivation any contradiction. This will allow us to deduce \( \neg \neg A \), and therefore \( A \). Let’s see an example to check how it works:

1) \( p \rightarrow q \)  
2) \( \neg q \)  
3) \( p \)  
4) \( q \)  
5) \( q \land \neg q \)  
6) \( \neg p \)

If we review the premises with care, and try to deduce the conclusion from them using a combination of the seven rules explained before, we will see that it is impossible to reach it (try it if you want). For that reason, we decide to use the RI \( \neg \). To do it, we start supposing the opposite of the conclusion that we want to deduce. Given that the conclusion is \( \neg p \), its negation is \( \neg \neg p \), which is the same as \( p \). Once we have done this, we should try to get a contradiction; that is, the simultaneous affirmation of a proposition and its opposite. In the example, we have done it in line 5, joining \( q \) and \( \neg q \). Once we have get a contradiction, and since logic detests contradictions and tries to escape from them, we must close the supposition. It is as if, trying to follow a path, we had realized that it is not possible to go further. After closing the supposition, in the next line we must always write the negation of our starting point, that is, the negation of the supposition that lead us to the contradiction. Since our supposition was \( p \) (line 3), its negation will be \( \neg p \). To annotate our proceeding, we will write the name of the rule we have used (RI \( \neg \)) followed by two numbers; the first one refers to the line where the supposition starts; while the second one refers to the last line of the supposition – which must be, besides, where the contradiction appears.

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4. STRATEGIES FOR DEDUCTION BY MEANS OF BASIC RULES

1. The principle of economy: sometimes, it is possible to get to a conclusion in many different ways but, although all of them might be correct, we must try to do it in the least number of steps.

2. Application of the rules that do not introduce subsidiary suppositions, with the exception of the RE $\lor$: when trying to do a deduction, we must always keep in mind both the conclusion and the initial premises, and try to reach the conclusion applying directly the rules that do not introduce subsidiary suppositions, with the exception of the cases in which in the premises or in some line of the derivation we have a disjunction that must be eliminated, in which case we should apply the RE $\lor$.

3. Application of RI $\rightarrow$: If there is an implication in the conclusion as main connective, the most convenient is to use RI $\rightarrow$. To apply it, we will always suppose the antecedent of the conclusion's main implication. If there are more implications in the conclusion, we should suppose, one by one, each of their antecedents, in the same order as their logical reach – first, the one with a greater reach; afterwards, the next one; and so on... – until we are left with the consequent alone. We should try to deduce such consequent applying the rules studied before.

4. Application of RI $\neg$: whenever we see that it is not possible to get to the conclusion applying the previous strategies, we should try to do it using RI $\neg$.

5. Derivation with no premises or initial suppositions: in this case, we must necessarily introduce subsidiary suppositions until we get the conclusion, having in mind, when choosing those suppositions, the rules of RI $\rightarrow$, and RI$\neg$. 
6. CONCLUDING REMARKS

Once explained both the rules of Natural Deduction and the basic strategies for the deduction of formulae, it is possible that someone might feel lost about what he or she is exactly trying to do here. After all, we could think, the aim of logic is to avoid errors in our reasoning, and after all that has been explained, it looks as if we had forgotten, completely, about the arguments of which we talked about at the beginning of the exposition. This, nevertheless, is not necessarily true. We must forget for some time about the “materiality” of arguments and focus only in their logical form to avoid going wrong. In any case, and as a conclusion, we will develop now a concrete example. In it, we will formalize in first place an argument from ordinary language, applying the needed rules to reach the conclusion and, at the same time, we will try to translate back into ordinary language each of the steps that we take. Let’s go:

1. **Argument:**
   “I study logic because I want to pass the exam and learn to reason. If I study logic, then I will be able to analyse the political discourses and discover when they are deceiving me. If I discover when they are deceiving me, then I will become more critical and will also mature. It is true that I want to pass the exam and that I do not want not to learn to reason. Therefore, I will mature.

2. **Formalization:**
   In this argument, as in any other one, everything that comes before “therefore” are its premises. The proposition after “therefore” is the conclusion. Having this in mind, we will start translating the statements in the following way:
   - \( p \) = I study logic
   - \( q \) = I want to pass the exam
   - \( r \) = I want to learn to reason
   - \( s \) = I will be able to analyse the political discourses
   - \( t \) = I will be able to discover when they are deceiving me
   - \( l \) = I will become more critical
   - \( m \) = I will mature

   Once we have translated the atomic propositions into our logical language, we can now translate the molecular ones as well, using the logical connectives. Each numbered line will correspond with a statement in the argument; that is, with the statement between a period and another period.: 
   1) \((q \land r) \rightarrow p\)  
   2) \(p \rightarrow (s \land t)\)  
   3) \(t \rightarrow (l \land m)\)
3. Demonstration applying the rules of Natural Deduction:

1) \((q \land r) \rightarrow p\)
2) \(p \rightarrow (s \land t)\)
3) \(t \rightarrow (l \land m)\)
4) \(q \land \neg \neg r\)
5) \(\neg \neg r\)
6) \(r\)
7) \(q\)
8) \(q \land r\)
9) \(p\)
10) \(s \land t\)
11) \(t\)
12) \(l \land m\)
13) \(m\)

4. Translated demonstration of the application of the rules of Natural Deduction:

5. In premise 4, it is stated that “It is true that I want to pass the exam and that I do not want not to learn to reason”; therefore, if this is true, then it is also true only a part of the sentence: it is true that I do not want not to learn to reason.

6. But “it is true that I do not want not to learn to reason” is in turn a double negation, that is equivalent to saying: I want to learn to reason.

7. As above, if premise 4 is true, then it is also true the other part of the statement: I want to pass the exam.

8. Therefore, I can now join both simple statements of line 6 and line 7, using a conjunction. In another way; if it is true that, from one side, I want to learn to reason; and from another, that I want to pass the exam, it is also true that: “I want to pass the exam and to learn to reason.”

9. But in our first premise we stated that: “I study logic because I want to pass the exam and to learn to reason”, or, in another way: “If I want to pass the exam and learn to reason, then I study logic”. We have already seen in step 8 that the antecedent of this statement is true: “it is true that I want to pass the exam and learn to reason”. Therefore, we can affirm the consequent: it is also true that “I study logic”.

10. In premise 2, we held that: “If I study logic, then I will be able to analyse the political discourses and learn when they are deceiving me”. We have already seen that it is true that “I study logic”, which is, in turn, the antecedent of this implication. Therefore, it will also be true its consequent: “I will be able to analyse the political discourses and learn when they are deceiving me”.

11. In the same way that described in step 5, if the whole conjunction in step 10

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is true, then each of its components are also true. Therefore, we can simply state
that it is true that: “I will be able to discover when they deceive me”.
12. But in premise 3 we stated that: “If I discover when they are deceiving me,
then I will become more critical and will mature”. Therefore, if as seen in step
11, the antecedent is true: “I discover when they are deceiving me”, the
consequent will also be true: “I will become more critical and will mature”.
13. And, in last place, and in the same way that in steps 5 and 10, if the
conjunction is true – as we have seen in step 12 –, one of its part will also be
true. Therefore, we can at last conclude that “I will mature”, which is the
conclusion we were trying to deduce.

5. Simplified demonstration:
5. Given premise 4, then I don't want not to learn to reason.
6. Given 5, then I want to learn to reason.
7. Given premise 4, then I want to pass the exam.
8. Given 7 and 6, then I want to pass the exam and learn to reason.
9. Given 8 and 1, then I study logic.
10. Given 2 and 9, then I will be able to analyse the political discourses and will
discover when they are deceiving me.
11. Given 10, then I will discover when they are deceiving me.
12. Given 3 and 11, then I will become more critical and will mature.
13. Therefore, and given 12, then I will mature.
7. TRUTH TABLE METHOD

We have already learned how to formalize an argument and how to prove whether the conclusion is deduced or not from the premises that we accept as true. But, we must give another step. The reason is that, as we have had the opportunity to see, the method of Natural Deduction has an extrinsic limitation, but non the less, very important. If we reflect carefully about its application, it basically depends on the capacity or skill of the person developing the deduction. What happens if it is too difficult? What happens if after many attempts we are not able to deduce the conclusion we are trying to deduce? According to the consequences of the method of Natural Deduction, we should think that the argument we are analysing is wrong, that the proposed conclusion doesn't follow from the premises... And, nevertheless, we could have look for a solution too little or too bad... or it could be simply that, to reach it, we would have to follow so long or so complex deductive chains that our reasoning powers wouldn't be able to meet the challenge.

This kind of doubt is very dangerous in logic. We are looking in logic for absolute certitude and therefore, we cannot dismiss an argument or reasoning as invalid if there is no absolute certitude that it is so, that the conclusion doesn't necessarily follow from the premises.

This is just the empty space that tries to occupy the Truth Table Method developed by Wittgenstein. It is a purely mechanic method, a characteristic that is, somehow now, an advantage. Because for being mechanical its success does not depend any more on our intelligence, but on spending enough time applying it – either personally or automatising it so that a computer would apply it for us. Through it, it is possible to decide efficiently and in a finite number of steps whether any argument or proposition is a tautology, contingent or a contradiction, which are the three possible results it yields. But it also has a basic or intrinsic limitation: it is only a method of decision; that is, it doesn't show us how, actually, we can deduce the conclusion; it only shows whether the conclusion is deduced necessarily from the premises, or if it is not possible to deduce it absolutely, or if it is sometimes deduced and some other times no. Put another way, the Truth Table Method is not a deductive method, it doesn't deduce anything, it only says: this proposition or argument (understood as the set of premises and conclusion) is a tautology, contingent or a contradiction.

Before getting into its explanation it is important to understand its sense – that is, precisely, what distinguishes us from machines. The Truth Table Method is based on a
fundamental logical principle stated by Aristotle in the 3rd century BC: the principle of bivalence. According to it, “every declarative statement is either true or false, but not both things at the same time and in the same sense”. That is, given any proposition, and independently of what its states (its materiality), we can completely be sure that it is either true or false, but no the two things at the same time.

This is, of course, quite obvious. But it wasn't until Wittgenstein entered the philosophical scene – in the 20th century – that its meaning became completely understood. When Wittgenstein starts his long philosophical path, Logical science had already reached its highest level of formalization. That is, it had already proceeded without any reference to the world of facts, and had developed a completely artificial language that “translated” every single element of any argument. Human reasoning itself had been formalized without any extrinsic interference. But this implied a very serious problem for the validity of the classical – aristoelian – principle of bivalence: if we do not take into account any reference to facts, it is impossible to know whether a statement or proposition is true or false, whether an argument – a set of propositions – is finally true or false. It is precisely here where Wittgenstein turns upside down the way to understand and apply the principles: it is true that after following the path of formalization and cutting any reference to the world of facts we can no longer know if a logical proposition is true or false; but we do know that, in as much as it is formalized and a priori, there are only two possibilities of truth for it and only two: it is either true or false.

For example, let \( p \) be any proposition. Without knowing anything about it, without knowing what it actually states, what it actually tells us about the world of facts, I do know that it is either true or false, but no both things at the same time. Therefore:

\[
\begin{array}{c|c}
 p & T \\
 F & \end{array}
\]

If we think carefully, Wittgenstein has only been coherent with the progressive formalization of Logic: if there is no reference to the world of facts, the only thing that we know about the truth or falsity of any proposition is its possibilities of truth or falsity, and not that actually it might be one thing or another.

The problem is relatively simple for an atomic proposition like the one in the example. What about molecular propositions – or about arguments that are equivalent to complex molecular propositions in the last resort? In these cases, the mechanism will be the same, although it might be more complex. A molecular proposition is a set of
atomic propositions. Therefore its possibilities of truth will depend, in first place, on
the possibilities of truth of its atomic propositions. But, since the atomic propositions
are linked by connectives (conjunction, disjunction, implication...) to build up the cited
molecular proposition, its possibilities of truth will also depend on the possibilities of
truth of those connectives. Let’s see an example.

Let $p \land q$ be a molecular proposition. Leaving aside for now the link between the
two atomic propositions (the conjunction), and given that we do not know its reference
to the world of facts, in order to know the possibilities of truth of such molecular
proposition, we need to determine all the possible combinations of truth and falsity for
the atomic propositions $p$ and $q$. Thus, it is possible for both of them to be true at the
same time; that the first one might be true and the second one false; the first one false
while the second one true; and, finally, that both of them might be false. In another
way:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$\land$</td>
<td>$q$</td>
</tr>
<tr>
<td>1$^{st}$</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>2$^{nd}$</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>3$^{rd}$</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>4$^{th}$</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

On a close look at the table, we have just established all the possible
combinations of truth and falsity for a molecular proposition. But we still don't know
the possibilities of truth of such molecular proposition (what is technically called its
truth value). To know it, it will be necessary to take into account the definition of truth
of the connective that links both atomic propositions or, in a more precise terminology,
the truth conditions of the connective that links both atomic propositions.

This is so because every connective, given its definition, implicitly states the
conditions that makes it true or false, leaving aside, as well, any reference to the world
of facts, to the materiality of its propositions – once again, another consequence of
formalization. For the conjunction, its conditions of truth state: a conjunction is true if
its components are true, and is false in any other case. It must be underlined that the
definition of the conditions of truth for the conjunction has been done without taking
into account, at any moment, the materiality of its components; that we have only taken
into account the possibilities for the combination of the possible truth values of its
components.

If we go back to our example, then we find that if both propositions ($p$ and $q$) are
true at the same time, the conjunction will be true; and that, in the rest of combinations, the conjunction will be false. Therefore:

<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>∧</th>
<th>q</th>
</tr>
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<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt;</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt;</td>
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<td>3&lt;sup&gt;rd&lt;/sup&gt;</td>
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<td>T</td>
</tr>
<tr>
<td>4&lt;sup&gt;th&lt;/sup&gt;</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

We have there, in the centre of the Truth Table – and in bold letters -, the possibilities of truth of a molecular proposition. As seen, we still hold the principle of bivalence: it is either true or false. The only thing that has been done is to make explicit the possibilities of being one thing or another, because, as we have repeated many times, we don’t know its reference to the world of facts.

Once we have understood this, and to end with the presentation of the capabilities of this method, it is convenient to say one more thing. After applying this method to the different propositions and arguments, we will find that the truth values of some molecular propositions are sometimes truths and some other falsities – like in our example. These propositions or arguments are contingent, because, from a logical point of view, from its mere formality, we cannot know for sure if they are necessarily true or necessarily false; the value depends on the possibilities of truth (or truth value) of the propositions that form them. But sometimes we will come across also, and as if by magic, with propositions or arguments whose final truth values will be, either all true (exclusively true) or all false (exclusively false). These are precisely the two types of propositions in which logic is specially interested in. We will now try to understand why.

What does it mean that, for a given proposition or argument, all its combinatory possibilities would be true? If we have in mind all that has been said until now, then that proposition would be always true, no matter the meaning of each atomic argument, no matter its reference to the world of facts: it is, if we prefer, absolutely true, true par excellence – although it might only be from a formal point of view, from the perspective of the structure of our thinking. These propositions are called, in logic, tautologies or tautological propositions, and are immune to falsity. They are of crucial importance for logic, because discovering propositions that are always true means discovering also the logical laws of our reasoning. As a matter of fact, for example, the rules for the transformation of formulae studied above are, all of them, logical laws once that we check their truth value.
With respect to those propositions that are always false, we are in the opposite case. They are absolute falsities, moods of reasoning we must avoid because they lead directly and only to error, to mistake. Logic has called them **contradictions**. With them, thought goes wrong, errs in the most absolute way. And whoever uses them will end contradicting himself in one way or another.

Taking all this into account, we will go on to explain, in a more systematic form, the ingenious Truth Table Method.

✓ It is a mechanical method to calculate the truth value of any proposition to know if it is a tautology, contingent, or a contradiction.

✓ To this aim, it consist of a mechanism that saves us from thinking, each time, in all the possible combinations of Truth/Falsity for a give proposition.

✓ Let’s use as example a molecular proposition composed of two variables. Leaving aside the connective that links both of them, it is clear that there are only four possible combinations of Truth/Falsity:

1. Both variables are true.
2. The first one is true and the other one false.
3. The first one is false and the other one is true.
4. Both are false.

✓ As explained, the possible combinations are four. We can also use the following law of the mechanism: **the number of rows (combinations) that compose a truth table so that all possible combinations are taken into account is equal to 2^n, where 2 is the number of truth values (true/false) and n the number of variables.**

✓ We already know the number of rows. How do we place the values to get all possible combinations?

✓ The procedure to place the values applies to how they are place in each column, that is, for each variable.

✓ Taking this into account:

1. For any variable, the number of values “true” is equal to 2^n/2; and the number of values “false” as well.
2. To place them, and for the first variable, we will write first all the values “true” and after all the values “false”.
3. For the second variable, we will place first **half** the total of values “true”, and after **half** the total of values “false”; next, the rest of the **half** of values “true” and the other rest of the **half** of values “false.
4. For the third variable, if there is any, we will write first **half of the half** of the total of values “true”, and after **half of the half** of the total of values
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“false”; next the following half of the half, and so on.
5. The procedure repeats itself as many times as needed.

✔ It must be taken into account that the procedure explained above applies to the variables, no matter the number of columns that they cause. That is: under the same variable, the same combination of truth values must be written.

✔ Once we have set this, we must proceed to solve the table, applying the logical definition of truth for each connective.

✔ In that resolution, we must always proceed from “inside to outside”; that is, from the connectives with less logical scope to those with more logical scope. Therefore, and in general, the order of resolution will be the following:

1. Negations applied to an atomic proposition.
2. Propositions inside of a parenthesis taking into account their hierarchy.
3. Negations of molecular propositions; that is, negations that apply to the contents of a parenthesis.

Connectives' Logical Definitions of Truth. Truth Conditions

The logical definitions of each connective that we've studied at the beginning of this unit include, as well, the conditions under which the link between two variables achieved by the connective is either true or false. These definitions of logical truth, or connective's truth conditions, are implied, as a matter of fact, in the immanent logic of our ordinary language; that is, they are part of the logical presuppositions of the ordinary language we use to argue or to convince someone or ourselves. With all this in mind, the truth conditions of each connective are the following:

(Note: given the importance of this method for computation, we will substitute the abbreviations True (T), False (F), for 1 and 0 respectively; therefore, True will be represented as 1; and false as 0.

1. Negation:

If a proposition is true, its negation is false; if it's false, its negation is true.

\[
\begin{array}{c|c}
\neg & p \\
\hline
0 & 1 \\
1 & 0 \\
\end{array}
\]

Ex: If it is true that “Cristiano R. is a football-player”, then it is false that “Cristiano R. is not a football-player” and vice-versa.

2. Conjunction:
A conjunction is true if all its components are true; and false in any other case.

<table>
<thead>
<tr>
<th>p</th>
<th>∧</th>
<th>q</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
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Ex.: If it is true that “Zubizarreta was a goal-keeper of FC Barcelona” and it is also true that “Zubizarreta was a goal-keeper of the Spanish National Football Team”, it is evident that it will also be true that “Zubizarreta was goal-keeper of FC Barcelona and of the Spanish National Football Team”. On the other side, if the first statement is true, but not the second; the conjunction of both would be false; as well as if the second is false and the first true. Finally, the conjunction of two false statements will also be false.

3. **Disjunction**: According to the type of disjunction.
   
   a) **Inclusive Disjunction**:
   
   An inclusive disjunction is true in every case, except when all its components are false.

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Ex.: The statement “Peter is football-player or baker” would be true in the following cases: Peter is both of them; Peter is one of them but not the other. It is only false, if Peter is neither football-player nor baker.

b) **Exclusive Disjunction**:

   An exclusive disjunction is true when only one of its two components is true (while the other is false); it is false in any other case, that is, when both are either true or false.
Ex.: The statement, “Mary was born in Madrid or Barcelona” is true only if Mary was born in Madrid, but not in Barcelona; or in Barcelona, but not in Madrid. It would be false if Mary was born in Madrid and in Barcelona (in this case, that is obviously impossible), or if Mary was not born in neither of the two cities.

4. **Conditional or implication:**
   An implication is true in every case, except when the antecedent is true and the consequent is false.

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Explanatory note: The truth conditions of this connective are, certainly, somehow difficult to understand and logicians have maintained passionate disputes over them through the history of this science. This is not the place to get deeply into them, but since the progressive use of mathematical symbolism in this logic that took place during the 19th and 20th century, the truth conditions of the implication have been defined as we have done above. This means that, from the perspective of nowadays logic, the implication (understood as the union of two propositions that are in a relation of conditionality) “If it is a sunny day, then I will go to the beach”, will be false only if it would be the case that it is a sunny day, but I don't go to the beach; because, in that case, from the truth of the condition (the antecedent), doesn't follow the truth of the consequent. This is the easy part to understand; in the same way that it is easy to understand that the
implication is true if it is the case the it is a sunny day and I go to the beach.

But, how can the truth of the implication in the other two cases be justified, that is, when the antecedent is false and the consequent true, or when both are false? The definition of truth for the implication holds that the antecedent $p$ is a sufficient condition for the consequent $q$, but not a necessary condition for $q$. This means that for any implication $(p \to q)$ to be true, if $p$ is the case, $q$ must also be the case; but the implication expressed in ordinary language by “if..., then...” doesn't mean that $q$ can only be the case if $p$ is the case, and in no other case. Continuing with the previous example, I can go to the beach if it is not a sunny day – for example, if I wish to see the sea.

Even more, it must be remembered that logic is bivalent (that is, that there are only two possible values of truth for each proposition – atomic or molecular –: either the expression is true or false). Thus, a conditional is also true if the antecedent is false and the consequent is false, because we must bear in mind that we are talking about the truth's definition of the connective; that is, whether or not the relation between both propositions expressed in the connective is true. When, in a conditional, the antecedent is false and the consequent true, we must not be distracted, when trying to understand the truth-state of the conditional, by the falsity of the antecedent. Let see an example: “If I am the King of Spain, then I was born in Salamanca”. Is the complete sentence true or false? Although it is obvious that I am not the King of Spain, and that – for those that know me – I was actually born in Salamanca, when deciding on the question of its truth, we must think on the complete proposition, on the relation “if..., then...” that links both parts of the sentence. Thus, it doesn't matter that the antecedent is false; it is not a necessary condition for the consequent's truth; in this case, it's only important the truth of the consequent, which is what makes truth the whole proposition: it is because I was born in Salamanca that the whole sentence is true. Something similar takes place in the last case. The conditional is still true although both its antecedent and consequent are false. In this case, ordinary language provides us with excellent examples like: “If you have been in the Everest, I am the king of Spain”. The conditionality (the relation between both propositions) that expresses the complex statement cited is true, no matter that both simple statements are false and just because they are false.

5. Biconditional or Equivalence:
A biconditional is true when its two components are true or false at the same time, and false in any other case, that is, when one is true and the other is false.
and vice-versa.

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Ex.: The biconditional “If and only if I know my Prince Charming, I will marry” is true if it is the case that I have known a Prince Charming and that I got married, or if neither I have known a Prince Charming nor I got married; and false in the rest of cases: if knew one, but didn't marry; or if I didn't know anyone, but I got married.

Example of the application of the Truth Table Method

Decide whether the following proposition is a tautology, contingent, or a contradiction:

\[
\{[p \rightarrow (q \land r)] \land \neg(q \land r)\} \rightarrow \neg p
\]

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IES HAYGON – 1º BACHILLERATO
8. FALLACIES, FALSE ARGUMENTS AND PARADOXES

Until now we have focus mainly on analysing the logical correction of our reasoning. But, it is as important to know when an argument is correct because it follows the logical rules of deduction, as it is to know the most common ways in which our reasoning goes wrong – sometimes unwillingly, but some others, on purpose. It is possible to distinguish two main erroneous ways of reasoning: fallacies or false arguments, and paradoxes. Let’s see now the definition of each as well as their main types.

8.1 Fallacies or False Arguments

In general, a fallacy is defined as any argument which, although seemingly correct, leads to a false conclusion. There are two main types of fallacies: formal fallacies and informal fallacies.

**Formal fallacies** are those that are not born out of the meaningful content of the propositions (their *materiality*), but from the logical structure of the argument (their *logical form*); that is, out of the validity or invalidity of the argument, independently of their apparent truth or falsity. The main formal fallacies are the following:

a) **Fallacy of the affirmation of the consequent:**
   We fall into this fallacy if after stating a conditional (if..., then...), and showing the consequent to be the case, we conclude that the antecedent is also true. In this case, the valid rule of the implication has been inverted for, according to it, if the antecedent is the case we can conclude the truth of the consequent. This means that in conditional propositions, from the truth of the consequent we cannot conclude neither the truth nor the falsity of the antecedent. The logical schema of this fallacy is the following:

\[ p \rightarrow q \]
\[ q \]
\[ p \]

Ex.: “If it is raining, then we get wet. We get wet. Therefore, it is raining”. It is an invalid way of reasoning, because taking into account the condition stated in the proposition, from getting wet we cannot necessarily conclude that it is raining; it could be that the fifth-floor neighbour would be watering her plants...

b) **Fallacy of the negation of the antecedent:**
   We fall into this fallacy whenever, after stating a conditional, its antecedent is
shown not to be the case, and then we conclude that the consequent is also false. The error, in this case, lies in believing that the negation of the antecedent implies the negation of the consequent. The simplest proof to see that this is invalid comes from remembering that every valid argument or logical law is, in last term, a tautology. If we check the truth value of the logical schema of this fallacy using the Truth Table Method we will see that it is not a tautology, but a contingent formula: sometimes it is true and some others is false.

\[
p \rightarrow q \\
\neg p \\
\neg q
\]

Ex.: “If it is raining, then we get wet. It is not raining. Therefore, we will not get wet”. In this case, the conclusion doesn't follow necessarily from the premises because although it might not be raining, it is equally possible to get wet or not to get wet: once again, the fifth-floor neighbour and her plants...

c) Fallacy of the conversion of the conditional:

We fall into this fallacy whenever, after stating a conditional, we conclude that the relation between the antecedent and the consequent held in it, is also true in the inverse sense: form the consequent to the antecedent. The error in this case lies in treating the conditional as a biconditional. Once again we could use the Truth Table Method to check the inaccuracy of this way of reasoning. Its logical schema is the following:

\[
p \rightarrow q \\
q \rightarrow p
\]

Ex.: “If it is raining, then we get wet. Therefore, if we get wet, then it is raining”. In this case, it is invalid to infer from the truth of the first conditional, the truth of the second one.

d) Fallacy of the negation of the antecedent and the consequent:

We fall into this fallacy whenever, after stating a conditional, we conclude that it is also true a conditional that relates the negation of the first antecedent and the negation of the first consequent. We can also check the error in this case using the Truth Table Method. Its logical schema is the following:

\[
p \rightarrow q \\
\neg p \rightarrow \neg q
\]

Ex.: “If it is raining, then we get wet. Therefore, if it is not raining, then we will not get wet”. In this case, it is invalid to infer that if the first implication would be true, the second one would also be true. Once more, the fifth-floor neighbour and her plants is playing with us...

e) Fallacy of the affirmation of an alternative:

This is one of the common fallacies of the disjunction. Given a disjunction, and once shown the truth of one of its terms, we conclude that the other term is false.
and is not the case. The error is discovered, in this case, using the Truth Table Method which will show us that this is a contingent formula. Under it, we think that in a disjunction, from the truth of one of its members follows the falsity of the other member. Its logical schema will be the following:

\[ p \lor q \quad \frac{p}{\neg q} \]

Ex.: “Peter is a baker or a football-player. He is a baker. Therefore, he is not a football-player”. As we can see, from being a baker we cannot conclude neither that he is a football-player nor that he is not.

**Informal fallacies** are those that proceed from the meaning of the words or of the affirmations or negations in an argument. They are, in short, arguments that seem correct, but after close inspection, finally end being erroneous. The main ones are the following:

a) **Fallacy of the conversion of an affirmative universal:**

It lies in concluding, from an affirmative universal proposition (one in which we predicate a quality or property of all the members of a set), another universal proposition in which we invert the terms (the subject becomes the predicate and vice-versa). This is the case in the following argument: “All physicists are scientists. Therefore, all scientists are physicists”. In the starting premise, we hold that the set of physicists is included in the set of scientists that is, therefore, more extensive. But in the conclusion of the converse, we hold illegitimately more than we stated in the first statement, because we know hold that it is the set of scientists the one included in the set of physicists. Therefore, from the truth of the first one we cannot deduce the truth of the second one.

b) **Fallacy of induction, of false generalization or of the accident:**

It lies in concluding illegitimately a general rule or universal statement out of a single or a plurality of particular cases. What is illegitimate here is to pass from the attribution in a concrete number of cases to the attribution in all the cases. This is the case in every argument in which we generalize from the observation of an insufficient number of cases, like when after seeing in the news that different immigrants have committed some crimes, we conclude: “Therefore, all immigrants are criminals”. From the fact that some might have committed crimes doesn't follow necessarily that all immigrants might be criminals, just as from the fact that some non-immigrants do not commit crimes doesn't follow that any non-immigrants are not criminals. This is, besides, the type of fallacy in which all people that guide their lives following prejudices fall into. A prejudice or bias is nothing more than an erroneous generalization: going from a few cases to all the cases. The only generalizations that are valid, strictly speaking, are
those reached at in an analytic form (for example: all triangles have three sides), or those in which it is actually proved that, case by case, something is the case. It is evident that this last possibility is quite problematic – briefly: how can we prove or check past cases or future cases?

c) **Fallacy of begging the question (petitio principii):**

It is the argument that takes as its premise, either implicitly or explicitly, precisely the conclusion that is to be proven. For example: “Abortion is the unjustified killing of a human being and as such is murder. Murder is illegal. So abortion should be illegal.” The conclusion of the argument is entailed in its premises. If one assumes that abortion is murder then it follows that abortion should be illegal because murder is illegal. Thus, the arguer is assuming abortion should be illegal (the conclusion) by assuming that it is murder. In this argument, the arguer should not be granted the assumption that abortion is murder, but should be made to provide support for this claim.

d) **Fallacy of the circular reasoning:**

It is the argument that pretends to support a conclusion using reasons which mean the same as what is intended to prove. For example: “Indeterminism must be true, because human beings are free.” In this case, indeterminism precisely holds that human beings are not determined, that they are free; so we are not proving anything at all; we are apparently giving a reason to hold the truth of indeterminism when, in fact, we are repeating what indeterminism says. Another example, widely cited, is: “Opium induces sleep because it has soporific properties.” The argument is fallacious because we have not actually said why opium induces sleep: stating that it is because it has soporific properties is like saying that it induces sleep because it has the property to induce sleep, precisely the meaning of being soporific.

e) **Fallacy of the false cause:**

This fallacy takes place when we consider as cause of an effect something which is not its cause, but that coincides or is previous to the effect. For example: “Roosters crow just before the sun rises. Therefore, roosters' crowing causes the sun to rise.”.

f) **Argument ad baculum:**

It is the fallacy committed when we do not use arguments to reason, but force, or coercion ranging from the physical threat (torture, slaps, ...) to the subtleness of appealing to one’s “own interest”. For example: “You should believe God exists because, if you don't, when you die you will be judged and God will send you to Hell for all of eternity. You don't want to be tortured in Hell, do you? If not, it is a safer bet to believe in God than to not believe.”

g) **Argument ad hominem:**
They are those arguments in which we refute someone's opinion, but not trying to show the falsity of his or her arguments, but discrediting the person that defends it. Arguments *ad hominem* are also considered those arguments that appeal to the circumstances around the person that defends an opinion to refute it. For example: “His arguments are false. What would you expect of a man that gets drunk every night?”

h) **Argument *ad autoritatem***:
Someone falls into this fallacy whenever a thesis is imposed as true, without any discussion, based on the someone's authority or power. For example: “Abortion is morally wrong and must be illegal. The Pope says so.”

i) **Argument *ad ignorantiam***:
A truth of a proposition is defended simply because its falsity has not been proven or vice-versa. For example: “Ghost really exists. Nobody has proven that they do not exist”. The problem here lies in that if we want to defend the truth or falsity of a given proposition, the weight of the proof is on our side: we must show ourselves why that proposition is true or false.

j) **Argument *ad misericordiam***:
It is the fallacy committed when pity or a related emotion such as sympathy or compassion is appealed to for the sake of getting a conclusion accepted. For example: “Oh, Officer. There's no reason to give me a traffic ticket for going too fast because I was just on my way to the hospital to see my wife who is in serious condition to tell her I just lost my job and the car will be repossessed.”

k) **Fallacy of equivocation**:
It is the fallacy committed when we use a word in the argumentation and the conclusion that has different meanings in one place and another. For example: “Criminal actions are illegal, and all murder trials are criminal actions, thus all murder trials are illegal”. In this case, the term “criminal actions” has two different meanings: the first one refers to an action that is criminal for being against the law, while the second one refers to the action of judging a criminal.

l) **Composition and division fallacies**:
The first one is committed when we illegitimately predicate of the whole what is correctly stated of one of its parts. For example: “Given that every natural number is finite, the set of natural numbers is also finite”. The second one is the inverse fallacy: we state of a part what is true of the whole: “Given that elephants are a species in peril of extinction, and that Ruka is an elephant; then Ruka is in peril of extinction”.

IES HAYGON – 1º BACHILLERATO
8.2. Paradoxes

A paradox, from the Greek words *para—doixas* ("against common opinion") is a surprising argument that questions the limits of our capacity to argue. It is common to distinguish between *logical paradoxes* and *semantic paradoxes*.

**a) Logical paradoxes**: They refer to formal arguments and to the logical properties of such arguments. The most famous might be the one called *Russell's paradox*, which he himself discover in set theory. Summarizing it, it states the following:

In set theory, sets are defined according to properties; thus, all objects or individuals that satisfy a given property belong to a given set; and those that do not, no. For example, set R could be defined by all objects that are red, so redness will be the property that defines the set. This kind of definition of sets, allows set mathematics to define sets whose members are not only individuals or objects (a red car, a red flower..), but also sets, for the only thing that we need to define a set is a property that determines the components that belong to the set. Russell's paradox refers to this last kind of sets: a set of sets. Let set R (now for Russell) be defined in the following way:

\[
\text{R is the set of all sets that are not members of themselves.}
\]

Before getting to understand where the paradox lies, one can wonder whether defining a set in such a manner makes sense; in another way: are there any sets that are members of themselves? Let us consider, as example, set I defined as the set of sets that have infinite members. There are, in theory, an infinity of sets that have infinite members, so set I is also infinite, and therefore; it is also member of itself! So it is not so absurd to classify the world of sets into two big sets: those that are members of themselves, and those that do not (our set R).

Then, where is the paradox? We discover it when we ask ourselves – in the same way that Russell did: where does set R belong to? Is set R a member of itself or is it not? Let us review both options:

1. If it is not a member of itself, set R belongs to set R that is defined by the property of “not being a member of oneself”. But then, R is member of R, which contradicts our starting point.
2. If R is a member of itself, then set R should not belong to set R. Once again we fall into a contradiction if we compare this last statement with the first one, because if the property that defines R is “not being a member of oneself”, and we suppose that R is a member of itself, R does
not belong to $R$, and then, it should be included in set $R$.

It is possible to express this same paradox in a way closer to common sense: let us imagine a library where there are two catalogues, one (catalogue A) where we include all those books of the library that do not refer to themselves in their own pages, and another catalogue (B) where we include all those books that do not refer to themselves in their own pages. Where should we include the second catalogue? If we write it down in the first one (A), then catalogue B wouldn't mention itself in its pages, and we should have included it in catalogue B, but not in A, where we wrote it. But if we write it down in catalogue B, then catalogue B mentions itself in its own pages, and we should have included it in catalogue A, but not in B.

b) **Semantic paradoxes**: they are born out of the meaning of the terms used in the arguments, especially of the meaning of the concepts of Truth and Falsity. The most famous is the paradox of the liar or Epimenides paradox:

“Epimenides, who is from Creta, says: 'all Cretans lie.”

Should we believe Epimenides?

If we analyse his statement, we find the following: if we consider true Epimenides' statement, then we are making false that same statement because Epimenides wouldn't be lying when he says that all Cretans lie – which is a contradiction. But if we consider false his statement, then we are confirming or making true what the statement says, and therefore, Epimenides is not lying but saying the truth – once again, we fall into a contradiction.

Another variant: if I tell you: “I am lying to you”, should you believe me? If you believe me, then you consider my statement true; but then, if my statement is true, I am lying to you when I say that I lie to you, so you shouldn't have believed me because I am lying. If you don't believe me, then you consider my statement false; but then, if my statement is false, I am saying the truth when I say that I lie to you, so you should have trusted me, because I was not lying.